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Correlation functions for the three-state superintegrable chiral Potts spin chain of finite lengths

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Abstract
We compute the correlation functions of the three-state superintegrable chiral Potts spin chain for chains of length 3, 4, 5. From these results we present conjectures for the form of the nearest-neighbor correlation function.

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1. Introduction

The free energy of the Ising model was first solved by Onsager [1] in 1944 who invented a method of solution based on what is now known as 'Onsager's algebra'. This algebra is generated from operators A_0 and A_1 which form a Hamiltonian

$$H = A_0 + \lambda A_1, \tag{1}$$

where

$$[A_0, [A_0, [A_0, A_1]]] = \text{const}[A_0, A_1] \tag{2}$$

and from A_0 and A_1 the full algebra is generated as [2]

$$\begin{aligned} &[A_j, A_k] = 4G_{j-k} \\ &[G_m, A_l] = 2A_{l+m} - 2A_{l-m} \\ &[G_j, G_k] = 0. \end{aligned} \tag{3}$$

For 41 years, the Ising model was the only known model which satisfied this algebra but in 1985 von Gehlen and Rittenberg [3] made the remarkable discovery that the Hamiltonian (1) with

$$A_0 = -\sum_{j=1}^{\mathcal{N}} \sum_{r=1}^{N-1} \frac{e^{i\pi(2r-N)/(2N)}}{\sin \pi r/N} Z_j^r Z_{j+1}^{\dagger r} \tag{4}$$

$$A_1 = -\sum_{j=1}^{\mathcal{N}}\sum_{r=1}^{N-1}\frac{e^{i\pi(2r-N)/(2N)}}{\sin\pi r/N}X_j^r, \tag{5}$$

where Z_j and X_j are direct product matrices

$$Z_j = I\otimes\cdots\otimes Z\otimes\cdots\otimes I \tag{6}$$

$$X_j = I\otimes\cdots\otimes X\otimes\cdots\otimes I, \tag{7}$$

where I is the $N\times N$ identity matrix, the $N\times N$ matrices Z and X are in the jth position in the product and have the matrix elements

$$Z_{j,k} = \omega^j\delta_{j,k} \tag{8}$$

$$X_{j,k} = \delta_{j,k+1} \tag{9}$$

with

$$\omega = e^{2\pi i/N} \tag{10}$$

also being a representation of Onsager's algebra (3). The Hamiltonian (1) with A_0 and A_1 given by (4) and (5) is called the N state superintegrable chiral Potts spin chain. When $N = 2$ the Hamiltonian of the superintegrable chiral Potts spin chain reduces to the Hamiltonian studied by Onsager [1].

It was discovered in [4] by means of explicit computations on chains of small length that the eigenvalues of the superintegrable chiral Potts Hamiltonian are all of the form

$$E = A + B\lambda + N\sum_{j=1}^{m}\pm(1+\lambda^2+a_j\lambda)^{1/2}. \tag{11}$$

This form was proven to follow directly from Onsager's algebra by [5–7]. The parameters a_j, A, B in (11) have been computed by the method of functional equations [8–11] where it is found, for λ suitably less than unity, that in the thermodynamic limit the ground-state energy per site is

$$e_0(\lambda) = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}E_0(\lambda;\mathcal{N}) = -(1+\lambda)\sum_{l=1}^{N-1}F\left(-\frac{1}{2},\frac{l}{N};1;\frac{4\lambda}{(1+\lambda)^2}\right), \tag{12}$$

where $F(a, b; c; z)$ is the hypergeometric function.

In this paper, we extend these finite chain computations from the eigenvalues of the Hamiltonian to the correlation functions in the ground state $\langle Z_0^r Z_R^{\dagger r}\rangle$.

There are several constraints that these correlations must satisfy. One such constraint is the obvious requirement that

$$\langle Z_0^{(N-r)}Z_1^{\dagger(N-r)}\rangle = \langle Z_0^r Z_1^{\dagger r}\rangle^*. \tag{13}$$

Furthermore in thermodynamic limit $\mathcal{N}\to\infty$, the correlation is related to the order parameter by

$$M_r^2 = \lim_{R\to\infty}\langle Z_0^r Z_R^{\dagger r}\rangle. \tag{14}$$

For $N = 2$, the order parameter is the spontaneous magnetization of the Ising model

$$M = (1-\lambda^2)^{1/8} \tag{15}$$

which was reported by Onsager [12] in 1948 and proven by Yang [13] in 1952. For the N state superintegrable chiral Potts spin chain, it was conjectured by Albertini *et al* [4] in 1988 and proven by Baxter [14] in 2005 that

$$M_r = (1-\lambda^2)^{r(N-r)/2N^2}. \tag{16}$$

The order parameter of the Ising model (15) may be computed [15] by the use of Szegő's theorem applied to the representation of the correlation function $\langle Z_0 Z_R^\dagger\rangle$ as a determinant [16] which is derived using free fermion methods first invented by Kaufman[17]. However, the chiral Potts order parameter (16) is computed by Baxter [14] by functional equation methods which do not extend to a computation of the correlation functions $\langle Z_0^r Z_R^{\dagger r}\rangle$.

Because the superintegrable chiral Potts model is a generalization of the Ising model with the same underlying Onsager algebra, there must be a structure of the Ising correlations which generalizes to superintegrable chiral Potts. However, because the Ising correlations have been computed by means of free Fermi methods and not by use of Onsager's algebra, this structure remains unknown.

Recently Au-Yang and Perk [18–21] and Nishino and Deguchi [22] have initiated the study of the eigenvectors of the superintegrable chiral Potts spin chain by the use of the Onsager algebra. These important studies are the necessary foundation for the computation of the correlation functions from the point of view of the Onsager algebra.

The purpose of this paper is to provide insight into the correlation functions of the superintegrable chiral Potts spin chain by explicitly calculating the correlations $\langle Z_0^r Z_R^{\dagger r}\rangle$ for the three state case $N=3$ for chains of finite length $\mathcal{N}=3,4,5$ which extends to correlation functions, the study of [4] of the ground-state energy $E_0(\lambda;\mathcal{N})$. For any value of $\mathcal{N}$, the nearest-neighbor correlations satisfy a sum rule coming from the ground-state energy $E_0(\lambda;\mathcal{N})$. This sum rule is presented and discussed in section 2. In section 3 we present the results of our finite chain computations and we conclude in section 4 with a discussion of the implications which the results for finite chains have for the correlations in the thermodynamic limit.

2. Sum rule

There is an elementary result known as Feynman's theorem that for any Hamiltonian which depends on a parameter λ that

$$\left\langle \frac{\partial}{\partial\lambda}H(\lambda)\right\rangle = \frac{\partial}{\partial\lambda}E_0(\lambda), \tag{17}$$

where $\langle O\rangle$ denotes the expectation value of the operator O in the ground state and $E_0(\lambda)$ denotes the ground-state energy as a function of λ. Therefore, it follows from (17) that for the superintegrable chiral Potts Hamiltonian (1), (4), (5), the expectation $\langle X_0^r\rangle$ satisfies the sum rule

$$\begin{aligned} -\mathcal{N}\sum_{r=1}^{N-1}\frac{\mathrm{e}^{\mathrm{i}\pi(2r-N)/(2N)}}{\sin\pi r/N}\langle X_0^r\rangle &= \frac{\partial E_0(\lambda;\mathcal{N})}{\partial\lambda} \\ &= B - N\sum_{j=1}^{m}\frac{\lambda+a_j/2}{(1+\lambda^2+a_j\lambda)^{1/2}} \end{aligned} \tag{18}$$

and $\langle Z_0^r Z_1^{\dagger r}\rangle$ satisfies

$$\begin{aligned} -\mathcal{N}\sum_{r=1}^{N-1}\frac{\mathrm{e}^{\mathrm{i}\pi(2r-N)/(2N)}}{\sin\pi r/N}\langle Z_0^r Z_1^{\dagger r}\rangle &= E_0(\lambda;\mathcal{N}) - \lambda\frac{\partial E_0(\lambda;\mathcal{N})}{\partial\lambda} \\ &= A - N\sum_{j=1}^{m}\frac{1+\lambda a_j/2}{(1+\lambda^2+a_j\lambda)^{1/2}}. \end{aligned} \tag{19}$$

3. Correlations for $N = 3$ and $\mathcal{N} = 3, 4, 5$

We will explicitly consider the three state case $N = 3$, where the Hamiltonian (1) is explicitly written as

$$H = -\sum_{j=1}^{\mathcal{N}}\left(\left(1-\frac{\mathrm{i}}{\sqrt{3}}\right)\left(Z_jZ_{j+1}^{\dagger}+\lambda X_j\right)+\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)\left(Z_j^2Z_{j+1}^{\dagger 2}+\lambda X_j^2\right)\right) \tag{20}$$

and using (13), the sum rule (19) is

$$\left(1-\frac{\mathrm{i}}{\sqrt{3}}\right)\left\langle Z_0Z_1^{\dagger}\right\rangle+\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)\left\langle Z_0Z_1^{\dagger}\right\rangle^* = -\frac{A}{\mathcal{N}}+\frac{3}{\mathcal{N}}\sum_{j=1}^{m}\frac{1+a_j\lambda/2}{(\lambda^2+1+a_j\lambda)^{1/2}}. \tag{21}$$

It is known from [10] that for small λ, the ground state is in the sector $P = 0$ and $Q = 0$, where P is the momentum of the state and

$$\mathrm{e}^{\mathrm{i}2\pi Q/3} = \prod_{k=1}^{\mathcal{N}} X_k. \tag{22}$$

For the ground state,

$$A = B = -P_a, \qquad \text{for} \quad \mathcal{N} \equiv -P_a \bmod 3 \quad P_a = 0, 1, 2 \tag{23}$$

and

$$a_j = \frac{2\left(1+t_j^3\right)}{1-t_j^3} \tag{24}$$

and t_j are the roots of the polynomial equation

$$\begin{aligned}0 = t^{-P_a}\{&(t-1)^{\mathcal{N}}(t\omega^2-1)^{\mathcal{N}}\omega^{-P_a}\\ &+(t\omega-1)^{\mathcal{N}}(t\omega^2-1)^{\mathcal{N}}+(t-1)^{\mathcal{N}}(t\omega-1)^{\mathcal{N}}\omega^{P_a}\}\end{aligned} \tag{25}$$

with $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$. From computations of Galois groups done on Maple, we see for $\mathcal{N} \leqslant 12$ that (25) can be explicitly solved in terms of radicals only for $\mathcal{N} = 3, 4, 5, 6, 9, 12$.

The eigenspace for the states with $P = Q = 0$ are of dimension 5 for $\mathcal{N} = 3$, dimension 8 for $\mathcal{N} = 4$ and dimension 17 for $\mathcal{N} = 5$. The correlations may now be computed in principle by computing the normalized eigenvectors in the subspace $P = Q = 0$ and then explicitly computing the matrix elements of $Z_0Z_R^{\dagger}$. In practice, the algebra is too formidable to do by hand and the computation must be computerized. When this is done and expressions are simplified by removing common factors, the results are as follows.

3.1. $\mathcal{N} = 3$

From (24) and (25), we find

$$9a^2 - 20 = 0 \tag{26}$$

and thus

$$a_1 = -\frac{2\sqrt{5}}{3}, \qquad a_2 = \frac{2\sqrt{5}}{3}. \tag{27}$$

Thus, defining

$$x_j = (\lambda^2+1+a_j\lambda)^{1/2}, \tag{28}$$

we find

$$\left\langle Z_0Z_1^{\dagger}\right\rangle = (1-\mathrm{i}\sqrt{3})\left(\frac{3}{40}-\frac{3\lambda^2-7}{40x_1x_2}\right)+\frac{3}{8}\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)\left(\frac{1+a_1\lambda/2}{x_1}+\frac{1+a_2\lambda/2}{x_2}\right). \tag{29}$$

From (29), we find

$$\left(1-\frac{\mathrm{i}}{\sqrt{3}}\right)\langle Z_0Z^\dagger\rangle = -\mathrm{i}\frac{4}{\sqrt{3}}\left(\frac{3}{40}-\frac{3\lambda^2-7}{40x_1x_2}\right)+\frac{1}{2}\left(\frac{1+a_1\lambda/2}{x_1}+\frac{1+a_2\lambda/2}{x_2}\right) \tag{30}$$

and hence the sum rule (19) is obviously satisfied. For $\lambda\to 0$, (29) is expanded as

$$\langle Z_0Z_1^\dagger\rangle = 1-\frac{2}{9}\lambda^2-\frac{17-\mathrm{i}3\sqrt{3}}{54}\lambda^4+O(\lambda^6) \tag{31}$$

and for $\lambda\to\pm\infty$,

$$\langle Z_0Z_1^\dagger\rangle = \frac{1}{3}\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)|\lambda|^{-1}+\frac{1}{6}(1-\mathrm{i}\sqrt{3})\lambda^{-2}+O(|\lambda|^{-3}). \tag{32}$$

3.2. $\mathcal{N}=4$

For $\mathcal{N}=4$,

$$27a^2+36a-20=0 \tag{33}$$

and thus there are again two roots a_j which are now given by

$$a_1=-\frac{2}{3}+\frac{4}{9}\sqrt{6},\qquad a_2=-\frac{2}{3}-\frac{4}{9}\sqrt{6}. \tag{34}$$

We define x_j as before (28) and obtain for $\langle Z_0Z_1^\dagger\rangle$ the result

$$\begin{aligned}\langle Z_0Z_1^\dagger\rangle &= \frac{1}{4}-(1-\mathrm{i}\sqrt{3})\frac{2\lambda-3}{40x_1x_2}\\ &\quad+(1+a_1\lambda/2)\,\frac{3}{320x_1}[36-\sqrt{6}+\mathrm{i}\sqrt{2}(3+2\sqrt{6})]\end{aligned} \tag{35}$$

$$\begin{aligned}&\quad+(1+a_2\lambda/2)\,\frac{3}{320x_2}\left[36+\sqrt{6}-\mathrm{i}\sqrt{2}(3-2\sqrt{6})\right]\\ &=\frac{3}{16}\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)+\frac{1}{16}(1-\mathrm{i}\sqrt{3})-(1-\mathrm{i}\sqrt{3})\frac{2\lambda-3}{40x_1x_2}\\ &\quad+(1+a_1\lambda/2)\,\frac{3}{320x_1}\left[30\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)+(6-\sqrt{6})(1-\mathrm{i}\sqrt{3})\right]\\ &\quad+(1+a_2\lambda/2)\,\frac{3}{320x_2}\left[30\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)+(6+\sqrt{6})(1-\mathrm{i}\sqrt{3})\right].\end{aligned} \tag{36}$$

We note that the coefficients of $1+\mathrm{i}3^{-1/2}$ and $1-\mathrm{i}3^{1/2}$ are both real, that

$$\begin{aligned}\left(1-\frac{\mathrm{i}}{\sqrt{3}}\right)\langle Z_0Z_1^\dagger\rangle &= \frac{1}{4}-\mathrm{i}\frac{1}{4\sqrt{3}}+\mathrm{i}\frac{1}{\sqrt{3}}\frac{2\lambda-3}{10x_1x_2}\\ &\quad+(1+a_1\lambda/2)\,\frac{3}{320x_1}\left[40-\mathrm{i}(6-\sqrt{6})\frac{4}{\sqrt{3}}\right]\\ &\quad+(1+a_2\lambda/2)\,\frac{3}{320x_2}\left[40-\mathrm{i}(6+\sqrt{6})\frac{4}{\sqrt{3}}\right]\end{aligned} \tag{37}$$

and thus the sum rule (19) is satisfied. For $\lambda\to 0$, (37) reduces to

$$\langle Z_0Z_1^\dagger\rangle = 1-\frac{2}{9}\lambda^2+\frac{1+\mathrm{i}\sqrt{3}}{162}\lambda^4+\frac{8\sqrt{3}\mathrm{i}}{729}\lambda^5+O(\lambda^6) \tag{38}$$

and for $\lambda \to \pm\infty$

$$\langle Z_0 Z_1^\dagger \rangle = \frac{1}{3}\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)|\lambda|^{-1} + \frac{1}{18}\left(1-\mathrm{i}\sqrt{3}\right)\lambda^{-2} + \frac{4}{81}(1-\frac{4\mathrm{i}}{\sqrt{3}})|\lambda|^{-3} + O(\lambda^{-4}). \tag{39}$$

We also compute $\langle Z_0 Z_2^\dagger \rangle$ and find

$$\langle Z_0 Z_2^\dagger \rangle = \frac{1}{4} - \frac{2\lambda - 3}{10x_1x_2} + (1+a_1\lambda/2)\frac{3(6-\sqrt{6})}{80x_1} + (1+a_2\lambda/2)\frac{3(6+\sqrt{6})}{80x_2}. \tag{40}$$

For small λ, this reduces to

$$\langle Z_0 Z_2^\dagger \rangle = 1 - \frac{2}{9}\lambda^2 - \frac{1}{81}\lambda^4 + O(\lambda^5) \tag{41}$$

and for $\lambda \to \pm\infty$

$$\langle Z_0 Z_2^\dagger \rangle = \frac{2}{9}\lambda^{-2} + \frac{20}{81}|\lambda|^{-3} + \frac{84}{729}\lambda^{-4} + O(\lambda^{-5}). \tag{42}$$

3.3. $\mathcal{N} = 5$

For $\mathcal{N} = 5$, there are three roots a_j that satisfy

$$81a^3 + 54a^2 - 228a - 88 = 0 \tag{43}$$

which are given by

$$a_1 = -\frac{2}{9} - \frac{2}{9}(W_+ + W_-) \tag{44}$$

$$a_2 = -\frac{2}{9} - \frac{2}{9}(\omega^2 W_+ + \omega W_-) \tag{45}$$

$$a_3 = -\frac{2}{9} - \frac{2}{9}(\omega W_+ + \omega^2 W_-), \tag{46}$$

where

$$W_\pm = \frac{1}{17}(7 \pm \mathrm{i}\sqrt{19})\left(\frac{5}{2}\right)^{1/3} w_\pm \tag{47}$$

with

$$w_\pm = (311 \pm \mathrm{i}9\sqrt{19})^{1/3}. \tag{48}$$

We define x_j as before (28) and obtain for $\langle Z_0 Z_1^\dagger \rangle$ the result

$$\begin{aligned}\langle Z_0 Z_1^\dagger \rangle = {} & \frac{3}{40}\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right) + \frac{37}{40\cdot 19}(1-\mathrm{i}\sqrt{3}) + \frac{9}{40}\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)\sum_{j=1}^{3}\frac{1+a_j\lambda/2}{x_j} \\ & + \frac{1}{2^3\cdot 5^2\cdot 19}(1-\mathrm{i}\sqrt{3})\sum_{j=1}^{3}\frac{91+b_j-27a_j/2-\lambda(67/3+7b_j/3-7\cdot 9a_j/2)}{x_j} \\ & + \frac{1}{2^3\cdot 5^2\cdot 19}(1-\mathrm{i}\sqrt{3})\sum_{1\leqslant j<k\leqslant 3}\frac{P_{j,k}}{x_jx_k},\end{aligned} \tag{49}$$

where

$$\begin{aligned} b_1 &= 1 + \left(\frac{5}{2}\right)^{1/3}(w_+ + w_-) \\ b_2 &= 1 + \left(\frac{5}{2}\right)^{1/3}(\omega^2 w_+ + \omega w_-) \\ b_3 &= 1 + \left(\frac{5}{2}\right)^{1/3}(\omega w_+ + \omega^2 w_-), \end{aligned} \tag{50}$$

and

$$\begin{aligned} P_{j,k} = 164 &+ \frac{81}{2}(a_j + a_k) + 7(b_j + b_k) \\ &+ \frac{\lambda}{3}\left[-56 + \frac{49\cdot 9}{2}(a_j + a_k) + 2(b_j + b_k)\right] \\ &+ \left(\frac{\lambda}{3}\right)^2[-504 - 13\cdot 81(a_j + a_k) - 9\cdot 13(b_j + b_k)]. \end{aligned} \tag{51}$$

It is easy to see that (49) satisfies the sum rule (19). When $\lambda \to 0$ (49) reduces to

$$\langle Z_0 Z_1^\dagger\rangle = 1 - \frac{2}{9}\lambda^2 + \frac{1}{162}(-9 + \mathrm{i}\sqrt{3})\lambda^4 + \frac{56}{729}\lambda^5 + O(\lambda^6) \tag{52}$$

and for $\lambda \to \pm\infty$,

$$\langle Z_0 Z_1^\dagger\rangle = \frac{1}{3}\left(1 + \frac{\mathrm{i}}{\sqrt{3}}\right)|\lambda|^{-1} + \frac{1}{18}(1 - \mathrm{i}\sqrt{3})\lambda^{-2} + \frac{4}{81}\left(1 + \frac{\mathrm{i}}{\sqrt{3}}\right)|\lambda|^{-3} + O(\lambda^{-4}). \tag{53}$$

4. Discussion

From the results (29), (36) and (49) for $N = 3$, we conjecture for arbitrary $\mathcal{N}$ that $\langle Z_0 Z_1^\dagger\rangle$ has the form

$$\langle Z_0 Z_1^\dagger\rangle = P_0 + \sum_j \frac{P_j(\lambda)}{x_j} + (1 - \mathrm{i}\sqrt{3})\sum_{j<k}\frac{P_{j,k}(\lambda)}{x_j x_k}, \tag{54}$$

where P_0 is a constant, $P_j(\lambda)$ is a polynomial linear in λ and $P_{j,k}(\lambda)$ is a polynomial at most quadratic in λ with real coefficients. In the limit $\mathcal{N} \to \infty$, the correlation $\langle Z_0 Z_1^\dagger\rangle$ will be a double integral. For $N = 2$, this correlation is a single integral and thus we expect that for general N, the correlation $\langle Z_0^r Z_1^{r\dagger}\rangle$ will be an $(N-1)$-fold integral.

We expect for arbitrary N that $\langle Z_0^r Z_R^{r\dagger}\rangle$ will be a $(N-1)R$ fold integral but the first evidence for this for $N = 3$ can only come $\mathcal{N} = 6$.

We also conjecture from the result (41) for $\langle Z_0 Z_2^\dagger\rangle$ and $\mathcal{N} = 4$ that for all even $\mathcal{N}$ the correlation for $N = 3$ $\langle Z_0 Z_{\mathcal{N}/2}^\dagger\rangle$ is real. In the limit $\mathcal{N} \to \infty$, we must have

$$\lim_{\mathcal{N}\to\infty}\langle Z_0^r Z_{\mathcal{N}/2}^{\dagger r}\rangle = (1 - \lambda^2)^{r(N-r)/N^2}. \tag{55}$$

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